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Path-integral quantization of the electromagnetic field in the Hopfield dielectric beyond dipole approximation

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Abstract

The paper contains an application of the path-integration method to the quantization of the electromagnetic field in a linear material medium modelled as the Hopfield dielectric. Except for the frequency dispersion of the medium, non-dipole effects are included leading to the wave vector dependence of the dielectric function. Starting from a local microscopic Lagrangian, the elimination of the matter degrees of freedom leads to the effective action describing dynamics of the classical field in the medium, from which the classical constitutive relation, non-local, both in time and space variables, could be determined. Full quantization of the model is achieved by integration over all fields with source terms included into the Lagrangian, and taking into account the constraint and gauge fixing term, characteristic for a quantized gauge theory. This gives the generating functional from which quantum propagators could be constructed. Operators of effective quantum electromagnetic and polarization fields were further retrieved from the propagators. The quantum constitutive equation containing the absorption noise term was also determined. The canonical equal time commutation rules are examined using both the BJL-limiting procedure and using explicit expressions for the field operators—the results in both cases are the same.

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1. Introduction

Path-integral quantization [1, 2] gives an alternative to the canonical quantization of classical mechanical and field theoretical physical systems. While it leads to an excellent and elegant interpretation of quantum mechanics based on the notion of classical trajectories, it also

appeared to be a very fruitful tool of the quantization of relativistic field theories, and especially the non-Abelian fields [3, 4]. In this paper the path integration will be used as a quantization method of the electromagnetic field in homogeneous and linear dispersive dielectric media, including not only the frequency dispersion, i.e. temporal non-locality of the constitutive equations, but also wave vector dispersion leading to spatial non-locality in the configuration space. The path-integral quantization of the electromagnetic field in a dispersive medium with only frequency dispersion was described in a previous publication [5].

Quantization of the electromagnetic field in dielectric materials has attracted attention of researchers for many years. First publications devoted to this subject [6–8] appeared soon after the formulation of quantum electrodynamics in vacuum. Theoretical efforts to understand the properties of the quantum electromagnetic field in polarizable matter, including dispersion and absorption, gained a strong momentum with the emergence of quantum optics and its various applications. One can distinguish two basic approaches that have been used for this purpose. In the macroscopic approach the matter degrees of freedom are accounted for in the form of material functions—electric permittivity (or dielectric function) in non-magnetic materials, assumed as the given function of frequency fulfilling the Kramers–Kronig relations. As examples of more recent publications in which various types of macroscopic approach were used one can mention [9–14]. In the phenomenological approach to the quantization of the electromagnetic field in an absorptive medium the losses are modelled by an additional 'noise current', which has to be introduced into the Maxwell equations for macroscopic fields. In the microscopic quantization the noise current is a result of the coupling of the medium polarization to reservoir degrees of freedom responsible for absorption [15].

The microscopic approach to quantization, where properties of matter on the atomic and molecular level are used as the starting point, is of a more fundamental nature than the macroscopic approach. At present the most frequently used microscopic model of the dispersive medium is that formulated by Hopfield in 1958 for dispersive and non-absorptive media [16], also known as the polariton model. The medium in the Hopfield model is described by the continuous harmonic oscillator field representing polarization of the medium, and the theory is quantized canonically. The canonical quantization scheme of Hopfield was generalized to an absorptive medium by Huttner and Barnett [15] giving the damped polariton model. The idea was to diagonalize the Hamiltonian of the system by constructing creation and annihilation operators of effective quanta—the polaritons. The polariton creation and annihilation operators were obtained from the original 'bare' operators by means of the Bogoliubov-type transformation. The damped polariton model was used to describe transient QED effects in dielectrics by Wubbs and Suttorp [17], where equations of motion were solved with the use of the Laplace transform. The Laplace transform method was further used by the same authors for the quantization of field in an inhomogeneous absorptive dielectric [18]. Recently the Huttner–Barnett model was generalized to the case of uniaxial crystalline dielectric media [19]. Another model, in which the medium is described by a set of harmonic oscillators, was also recently proposed [20]. In this model the continuum of oscillators' characteristic for the Hopfield model was replaced by a finite collection of harmonic oscillators.

Microscopic quantization of the electromagnetic field in dielectric media can also be based on different assumptions. In the papers by Juzeliūnas [21, 22], the discrete nature of matter on microscopic level was explicitly accounted for, with an arbitrary number of energy levels for each molecule. In another attempt to quantize the electromagnetic field in a dielectric medium, the matter degrees of freedom were modelled by a collection of two-level atoms [23].

The Huttner–Barnett Lagrangian [15] is used in this paper as a basic assumption of the model, with the only difference that terms containing spatial derivatives of the polarization

field have been included. This corresponds to including non-dipolar effects in the response of the medium to the electromagnetic field, and leads to spatial non-locality of the constitutive equations, or the wave vector dispersion of the dielectric constant. Whereas the dipole approximation, leading only to frequency dispersion, is equivalent to long wavelength approximation, including the effects beyond dipole approximation allows us to also consider waves with a shorter length. The suitable length scale in the present model will be given later.

As a first step, the matter degrees of freedom will be eliminated by functional integration over the oscillator fields representing the medium. As a result, one obtains the effective action of the macroscopic classical electromagnetic field, describing motion of the field in the material environment. The notion of the effective action in the present case is similar to the effective action in vacuum QED [24], which also results by the elimination of the matter degrees of freedom. In the case of vacuum QED they are described by electron–positron fields, and in the case of the damped polariton model these are the oscillator fields used to model the medium. While the notions of effective action in vacuum and dielectric QED are basically very similar, the technical side of the derivation is entirely different in both cases. The field equations of the present model are linear, so that the elimination of medium degrees of freedom is straightforward due to the Gaussian nature of all functional integrations. In vacuum QED, the elimination of the electron–positron field requires a more sophisticated method [24].

Full quantization requires functional integration over all fields with the source terms included into the microscopic action functional [25]. As a result of this procedure one obtains the generating functional from which the propagators of the fields can be calculated by functional differentiation with respect to the source functions [26]. The model is quantized in a way appropriate for a gauge theory, with the constraint (Gauss law) and the gauge fixing term accounted for. The propagators are basic objects in the path-integral quantization scheme of a field theory. On the other hand, for most physical applications of QED in dielectric media more convenient are the field operators constructed with the use of creation and annihilation operators of effective quanta—the polaritons [16]. It is shown in this paper that the effective field operators can be retrieved from the propagators. To make the derivation complete, the equal time commutation rules (ECTR) have been examined with the use of the limiting BJL procedure [27, 28]. It has been also shown that this leads to the same result as that expected from the 'naive' ECTR, derived directly from the Lagrangian.

In section 2 the microscopic Lagrangian density $\mathcal{L}$ based on the Huttner–Barnett expression [15] is given as the starting point for further consideration. It is also shown here how the terms leading to wave vector dispersion can be included into the original expression for $\mathcal{L}$. The effective action of the classical electromagnetic field is further calculated by the elimination of the matter degrees of freedom. Effective action is then used to find the frequency and wave vector dependent dielectric function, and the constituent equation for the classical $\mathbf{D}$ and $\mathbf{E}$ fields is derived. In section 3, which constitutes the main part of the paper, full quantization of the model is performed. The quantization is performed taking into account the subtleties connected with the constraint and the choice of gauge. The generating functional is used to calculate the propagators of the electromagnetic and polarization fields. In section 4 the ECTR are examined with the use of the BJL limit. As a next step, the operators of the effective quantum fields are retrieved from the propagators and the constitutive equation for these quantum fields is established. This is done in section 5. Section 6 contains final remarks and some details of the calculations are given in appendix A. In appendix B, it is shown that the ECTR discussed in section 4 can also be derived directly from the expressions for the field operators calculated in section 5.

2. Lagrangian density and effective action of the electromagnetic field in the medium

The Lagrangian density of a linear isotropic medium interacting with the electromagnetic field is in the long wavelength limit assumed in the form [15]

$$\mathcal{L} = \frac{\varepsilon_0}{2}\mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2 + \frac{1}{2\varepsilon_0\omega_0^2\chi_r}\left(\dot{\mathbf{P}}^2 - \omega_0^2\mathbf{P}^2\right) + \mathbf{P}\cdot\mathbf{E} + \int_0^\infty \mathrm{d}\omega' \left[\frac{\varrho}{2}\dot{\mathbf{Y}}_{\omega'}^2 - \frac{\varrho}{2}\omega'^2\mathbf{Y}_{\omega'}^2 - f(\omega')\mathbf{P}\cdot\dot{\mathbf{Y}}_{\omega'}\right]. \tag{2.1}$$

Here $\mathbf{P}$ is the polarization field, modelled by a harmonic oscillators with eigenfrequency ω_0, interacting with the electric field. χ_r denotes static polarizability of the medium and the term quadratic in the polarization field has been written here in the form similar to that used in [16]. The harmonic oscillator fields $\mathbf{Y}_{\omega'}$ with continuous distribution of eigenfrequencies correspond to the degrees of freedom responsible for absorption, and $f(\omega')$ gives the frequency-dependent coupling of these reservoir degrees of freedom to the polarization field. The parameter ϱ is the density of the reservoir oscillators per unit frequency interval. The polarization field, having a physical sense of the dipole electric moment per unit volume, interacts with the electric field via the $\mathbf{d}\cdot\mathbf{E}$ type of coupling.

The theory described by (2.1) leads after the elimination of the matter degrees of freedom [5, 15] to the macroscopic constitutive equation of the form

$$\mathbf{D}(t,\mathbf{r}) = \varepsilon_0\mathbf{E}(t,\mathbf{r}) + \varepsilon_0\int_{-\infty}^{t}\chi(t-t')\mathbf{E}(t',\mathbf{r}), \tag{2.2}$$

where $\chi(t)$ is the linear response function of the medium. This form of constitutive equation is characteristic for the dipole approximation in which the spatial dependence of the response function is disregarded. In a more exact treatment of the reaction of the dielectric medium to the external electromagnetic field, next terms, corresponding to higher multipoles, should be taken into account. This would then lead to the constitutive equation of the form

$$\mathbf{D}(t,\mathbf{r}) = \varepsilon_0\mathbf{E}(t,\mathbf{r}) + \varepsilon_0\int_{-\infty}^{t}\mathrm{d}t'\int \mathrm{d}^3\mathbf{r}'\,\chi(t-t',\mathbf{r}-\mathbf{r}')\mathbf{E}(t',\mathbf{r}'), \tag{2.3}$$

with time- and space-dependent response function. When higher multipoles are included into the constitutive equations, leading to the spatially non-local relation between the $\mathbf{D}$ and $\mathbf{E}$ fields, the polarization field can no longer be considered as a dipole polarization. It has to be considered rather as a generalized polarization [29] with the interaction term of the same form as in (2.1).

In the Huttner–Barnett model, with the Lagrangian density (2.1), the absence of spatial non-locality, or wave vector dispersion, is a direct consequence of the fact that the Lagrangian density contains no spatial derivatives of the matter fields, and in particular of the polarization field $\mathbf{P}$. The simplest generalization of (2.1) to the form containing spatial derivatives of the generalized polarization field can be given for an isotropic medium as

$$\mathcal{L}(\mathbf{E},\mathbf{B},\mathbf{P},\mathbf{Y}_\omega) = \frac{\varepsilon_0}{2}\mathbf{E}^2 - \frac{1}{2\mu_0}\mathbf{B}^2 + \frac{1}{2\varepsilon_0\omega_0^2\chi_r}\left(\dot{\mathbf{P}}^2 - v^2\frac{\partial P_i}{\partial x_j}\frac{\partial P_i}{\partial x_j} - \omega_0^2\mathbf{P}^2\right) + \mathbf{P}\cdot\mathbf{E} + \int_0^\infty \mathrm{d}\omega' \left[\frac{\varrho}{2}\dot{\mathbf{Y}}_{\omega'}^2 - \frac{\varrho}{2}\omega'^2\mathbf{Y}_{\omega'}^2 - f(\omega')\mathbf{P}\cdot\dot{\mathbf{Y}}_{\omega'}\right], \tag{2.4}$$

where summation over repeated vector indices is understood. The parameter v has a dimension of velocity and its physical meaning will be clarified below. Due to the assumed isotropic character of the medium the term containing spatial derivatives must be rotationally invariant. For simplicity no spatial derivatives of the reservoir field $\mathbf{Y}_\omega$ were included.

The form of the term with spatial derivatives of the polarization field is not unique. Another contribution consistent with the isotropic character of the medium could be of the form $-\left(2\varepsilon_0\omega_0^2\chi_r\right)^{-1}v_1^2\partial_i P_j\partial_j P_i$. As will be seen below, (2.4) leads to the dielectric tensor proportional to δ_{ij}, whereas it can be shown that the $\partial_i P_j\partial_j P_i$ term would give an additional term in the dielectric tensor proportional to $k_i k_j$. In both cases, the transverse part of the tensor, proportional to $\delta_{ij}-k^{-2}k_i k_j$, is the same and they differ by the form of the longitudinal part $\sim k_i k_j$. In general, transverse modes of the electric field in dielectric media are of principal interest [30]. For this reason, and also in order to keep the number of phenomenological parameters as small as possible, no additional derivative terms will be included into the Lagrangian density.

To give an estimation of the parameter v, note that wave vector dispersion can be disregarded when the wave vector k is small, i.e. for wavelengths large in comparison with the characteristic length scale of the model. A typical parameter with the dimension of length in the present model is given by v/ω_0. Therefore, a characteristic scale of wave vectors is ω_0/v, and long wavelength approximation, i.e. only frequency dispersion in the constitutive equations, is justified for $k \ll \omega_0/v$. On the other hand, response of the medium to the electromagnetic field can be described in the dipole approximation for wavelengths larger than the interatomic or intermolecular distance, which in a solid is typically of the order of 10^{-10} m. One may therefore expect the dipole approximation to fail for wavelengths, $\lambda = 2\pi/k$, of this order of magnitude. The length scale in the present model is given by v/ω_0 and, although no intermolecular distance appears in the microscopic Lagrangian density (2.4) explicitly, to obtain a physical criterion of applicability of the dipole approximation it is reasonable to assume that both length scales are comparable, i.e. $v/\omega_0 \sim 10^{-10}$ m. Assuming resonance frequency in the optical region, $\omega_0 \sim 10^{15}-10^{16}$ Hz, one obtains $v \sim 10^5-10^6$ m s^{-1}, i.e. $10^{-3}-10^{-2}$ of the velocity of light in vacuum.

The basic object in the path-integral formulation of an effective field theory is the ground state persistence amplitude (vacuum-to-vacuum probability amplitude),

$$C[\mathbf{E},\mathbf{B}] = \exp\left(\frac{\mathrm{i}}{\hbar}S_{\mathrm{eff}}[\mathbf{E},\mathbf{B}]\right) = \int [\mathrm{d}\mathbf{P}][\mathrm{d}\mathbf{Y}_\omega]\exp\left(\frac{\mathrm{i}}{\hbar}S[\mathbf{E},\mathbf{B},\mathbf{P},\mathbf{Y}_\omega]\right), \tag{2.5}$$

where S_{eff} is, by definition, the effective action [31], and $\int[\mathrm{d}\mathbf{P}][\mathrm{d}\mathbf{Y}_\omega]$ denotes functional integration over the matter fields. Functional integrations in (2.5) have to be performed over the classical fields vanishing for time $\to \pm\infty$. This boundary condition can be implemented by multiplying the right-hand side by ground state wavefunctions in remote past and remote future, and perform functional integration over all fields [32]. The appropriate expression reads

$$C[\mathbf{E},\mathbf{B}] = \int [\mathrm{d}\mathbf{P}][\mathrm{d}\mathbf{Y}_\omega]\exp\left(\frac{\mathrm{i}}{\hbar}S[\mathbf{E},\mathbf{B},\mathbf{P},\mathbf{Y}_\omega]\right)\Psi_0^{\mathrm{out}*}[\mathbf{P},\mathbf{Y}]\Psi_0^{\mathrm{in}}[\mathbf{P},\mathbf{Y}]. \tag{2.6}$$

The wavefunctions of a field system are represented by functionals of the fields. The expression of the type (2.6) for the ground state persistence amplitude results with the $-\mathrm{i}\epsilon$ term in the denominators of the propagator, giving a prescription for avoiding singularities of the propagators [32].

The ground state persistence amplitude is a functional of the external electromagnetic field and its modulus squared gives the probability that the medium remains in its ground state in future if it was in this state in the past, despite of the interaction with the electromagnetic field. S is the microscopic action functional

$$S[\mathbf{E},\mathbf{B},\mathbf{P},\mathbf{Y}_\omega] = \int \mathrm{d}t\int \mathrm{d}^3x\,\mathcal{L}(\mathbf{E},\mathbf{B},\mathbf{P},\mathbf{Y}_\omega). \tag{2.7}$$

The linear dielectric described by the microscopic Lagrangian density (2.1) without spatial derivatives of the polarization field has been treated with the help of the path-integration method in [5]. Functional integrations in (2.5) are Gaussian and can be done explicitly. The integration over the reservoir fields $\mathbf{Y}_\omega$ can be performed in exactly the same way as in [5], and leads to the ground state persistence amplitude in the form

$$C[\mathbf{E},\mathbf{B}] = \exp\left(\frac{\mathrm{i}}{\hbar}S_{\mathrm{em}}[\mathbf{E},\mathbf{B}]\right)\int[\mathrm{d}\mathbf{P}]\exp\left[\frac{\mathrm{i}}{2\hbar}\int\mathrm{d}t\int\mathrm{d}^3x\,\frac{1}{\varepsilon_0\omega_0^2\chi_r}\left(\dot{\mathbf{P}}^2 - v^2\partial_i P_j\partial_i P_j - \tilde{\omega}_0^2\mathbf{P}^2\right)\right.$$
$$\left. - \frac{\mathrm{i}}{\hbar}\int\mathrm{d}t\int\mathrm{d}^3x\,\mathbf{E}\cdot\mathbf{P}\right]\exp\left[\frac{\mathrm{i}}{2\hbar}\frac{1}{\varrho}\int\mathrm{d}t\int\mathrm{d}t'\int\mathrm{d}^3x\,\mathbf{P}(t,\mathbf{x})G(t-t')\mathbf{P}(t',\mathbf{x})\right], \tag{2.8}$$

where S_{em} is the action of the free electromagnetic field and $\tilde{\omega}_0$ is the shifted resonance frequency [15],

$$\tilde{\omega}_0^2 = \omega_0^2 + \int_0^\infty \mathrm{d}\omega'\,\frac{w(\omega')^2}{\varrho^2}, \tag{2.9}$$

with

$$w(\omega) = \left(\varepsilon_0\omega_0^2\chi_r\varrho\right)^{1/2} f(\omega). \tag{2.10}$$

The integral kernel $G(t)$ is given as

$$G(t) = \int_0^\infty \mathrm{d}\omega'\,\omega'^2 f(\omega')^2 D_F(t,\omega'), \tag{2.11}$$

where the propagator of the reservoir oscillators has the Fourier representation

$$D_F(t,\omega') = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{\exp(-\mathrm{i}\omega t)}{\omega'^2-\omega^2-\mathrm{i}\epsilon}, \tag{2.12}$$

with the $-\mathrm{i}\epsilon$ term in the denominator. The next step in the elimination of the matter degrees of freedom is the functional integration over the polarization field, which is also Gaussian and leads to the final form of the ground state persistence amplitude

$$C[\mathbf{E},\mathbf{B}] = \exp\left[\frac{\mathrm{i}}{\hbar}S_{\mathrm{em}}[\mathbf{E},\mathbf{B}] + \frac{\mathrm{i}}{2\hbar}\int_x\int_{x'}\mathbf{E}(x)\Gamma(x-x')\mathbf{E}(x')\right], \tag{2.13}$$

where a compact notation has been used in which x denotes the set of variables t and $\mathbf{x}$, and

$$\int_x f(x) = \int\mathrm{d}t\int\mathrm{d}^3x\,f(t,\mathbf{x}).$$

The integral kernel $\Gamma(x)$ is an inverse of the operator in the part of the exponent in (2.8) quadratic in the polarization field $\mathbf{P}$. It fulfils the integro-differential equation

$$\frac{1}{\varepsilon_0\omega_0^2\chi_r}\left(\frac{\partial^2}{\partial t^2} + \tilde{\omega}_0^2 - v^2\nabla^2\right)\Gamma(t-t',\mathbf{x}-\mathbf{x}')$$
$$-\frac{1}{\varrho}\int\mathrm{d}t''G(t-t'')\Gamma(t''-t',\mathbf{x}-\mathbf{x}') = \delta(t-t')\delta^{(3)}(\mathbf{x}-\mathbf{x}'), \tag{2.14}$$

where $G(t)$ is given by (2.11). The Fourier transform of the integral kernel $\tilde{\Gamma}(\omega,\mathbf{k})$ fulfils the algebraic equation

$$\left[\tilde{\omega}_0^2 - \omega^2 + v^2k^2 - \frac{\varepsilon_0\omega_0^2\chi_r}{\varrho}\tilde{G}(\omega)\right]\tilde{\Gamma}(\omega,\mathbf{k}) = \varepsilon_0\omega_0^2\chi_r. \tag{2.15}$$

The Fourier transform of the function $G(t)$ is given by

$$\tilde{G}(\omega) = \int_0^\infty \mathrm{d}\omega' \frac{\omega'^2 f(\omega')^2}{\omega'^2 - \omega^2 - \mathrm{i}\epsilon}, \tag{2.16}$$

and after using (2.9) and (2.10) can be written as

$$\tilde{G}(\omega) = \frac{\varrho}{\varepsilon_0 \omega_0^2 \chi_r} \left[\tilde{\omega}_0^2 - \omega_0^2 + \omega^2 \lambda_F(\omega)\right], \tag{2.17}$$

where [5]

$$\lambda_F(\omega) = \frac{1}{\varrho^2} \int_0^\infty \mathrm{d}\omega' \frac{w(\omega')^2}{\omega'^2 - \omega^2 - \mathrm{i}\epsilon}. \tag{2.18}$$

Substitution of (2.17) into (2.15) gives for $\tilde{\Gamma}$

$$\widetilde{\Gamma}(\omega, \mathbf{k}) = \frac{\varepsilon_0 \omega_0^2 \chi_r}{\omega_0^2 + v^2 \mathbf{k}^2 - \omega^2 - \omega^2 \lambda_F(\omega)}. \tag{2.19}$$

Due to the presence of the complex-valued function λ_F in the denominator poles of the integral kernel, $\widetilde{\Gamma}$ have been removed from the real axis to the complex ω-plane. It can be checked that the pole with $\mathrm{Re}\,\omega > 0$ is in the lower half-plane and that with $\mathrm{Re}\,\omega < 0$ in the upper half-plane. In the absorptionless limit, which corresponds to $\lambda_F \to 0 + \mathrm{i}\epsilon$, one gets the correct $-\mathrm{i}\epsilon$ prescription for avoiding the singularities,

$$\widetilde{\Gamma} \to \frac{\varepsilon_0 \omega_0^2 \chi_r}{\omega_0^2 + v^2 \mathbf{k}^2 - \omega^2 - \mathrm{i}\epsilon}. \tag{2.20}$$

It follows from (2.5) and (2.13) that the effective action of the electromagnetic field in a linear dispersive medium is given by

$$S_{\mathrm{eff}}[\mathbf{E}, \mathbf{B}] = S_{\mathrm{em}}[\mathbf{E}, \mathbf{B}] + \frac{1}{2} \int_x \int_{x'} \mathbf{E}(x) \Gamma(x - x') \mathbf{E}(x'). \tag{2.21}$$

Since $\tilde{\Gamma}^*(-\omega, -\mathbf{k}) \neq \tilde{\Gamma}(\omega, \mathbf{k})$, the kernel $\Gamma(t, \mathbf{x})$ is not real and as a consequence the effective action is complex. It can be shown that the imaginary part of the effective action is positive, so that the ground state persistence probability is smaller than unity. Interaction with the external electromagnetic field leads to the excitation of the medium which therefore may not remain in its ground state. The energy of this excitation is further absorbed by the reservoir degrees of freedom.

Effective action of the electromagnetic field can be used to find the constitutive equation and dielectric function of a dispersive dielectric. The constitutive equation, i.e. the relation between the displacement field $\mathbf{D}$ and the electric field $\mathbf{E}$, could be determined with the use of the general formula, which in the case of a local theory reads [33]

$$\mathbf{D}(t, \mathbf{x}) = \frac{\partial \mathcal{L}(\mathbf{E}, \mathbf{B})}{\partial \mathbf{E}(t, \mathbf{x})}, \tag{2.22}$$

where $\mathcal{L}(\mathbf{E}, \mathbf{B})$ is a local Lagrangian density. In the present case the effective action is complex and non-local in space and time, so that the general formula (2.22) cannot be directly applied. This difficulty can be overcome by expressing the effective action in terms of Fourier transforms of the electric field and of the integral kernel, and then calculating the displacement field by differentiating the effective Lagrangian with respect to the complex conjugate of the Fourier transform of the electric field [5]. The effective action (2.21) can be expressed by Fourier transforms of the integral kernel and the fields as

$$\begin{aligned} S_{\mathrm{eff}}[\mathbf{E}, \mathbf{B}] = &\int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \int' \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\varepsilon_0 |\mathbf{E}(\omega, \mathbf{k})|^2 - \frac{1}{\mu_0} |\mathbf{B}(\omega, \mathbf{k})|^2\right] \\ &+ \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \int' \frac{\mathrm{d}^3 k}{(2\pi)^3} \tilde{\Gamma}(\omega, \mathbf{k}) |\mathbf{E}(\omega, \mathbf{k})|^2. \end{aligned} \tag{2.23}$$

Since $\mathbf{E}(\omega, \mathbf{k})$ and $\mathbf{B}(\omega, \mathbf{k})$ are complex with $\mathbf{E}^*(\omega, \mathbf{k}) = \mathbf{E}(-\omega, -\mathbf{k})$ and $\mathbf{B}^*(\omega, \mathbf{k}) = \mathbf{B}(-\omega, -\mathbf{k})$, which doubles the number of variables, the integrations in the reciprocal space must be performed over the positive frequency semi-axis and the half space of the vector $\mathbf{k}$ [15] (the half-space $\mathbf{k}$-integration is denoted by prime). The effective Lagrangian density in the reciprocal space reads

$$\mathcal{L}_{\text{eff}}(\mathbf{E}, \mathbf{B}) = \varepsilon_0 |\mathbf{E}(\omega, \mathbf{k})|^2 - \frac{1}{\mu_0} |\mathbf{B}(\omega, \mathbf{k})|^2 + \tilde{\Gamma}(\omega, \mathbf{k}) |\mathbf{E}(\omega, \mathbf{k})|^2. \tag{2.24}$$

The displacement field $\mathbf{D}(\omega, \mathbf{k})$ in the reciprocal space will be calculated as [5]

$$\mathbf{D}(\omega, \mathbf{k}) = \frac{\partial \mathcal{L}_{\text{eff}}}{\partial \mathbf{E}^*(\omega, \mathbf{k})} = \varepsilon_0 \mathbf{E}(\omega, \mathbf{k}) + \tilde{\Gamma}(\omega, \mathbf{k}) \mathbf{E}(\omega, \mathbf{k}), \tag{2.25}$$

valid for $\omega > 0$. The dielectric function for $\omega > 0$ can be retrieved from this equation as

$$\varepsilon_+(\omega, \mathbf{k}) = 1 + \frac{1}{\varepsilon_0} \tilde{\Gamma}(\omega, \mathbf{k}) = 1 + \frac{\omega_0^2 \chi_r}{\omega_0^2 + v^2 \mathbf{k}^2 - \omega^2 - \omega^2 \lambda_F(\omega)}. \tag{2.26}$$

Analytic continuation of the dielectric function to negative values of frequencies is given by $\varepsilon_-(\omega, \mathbf{k}) = \varepsilon_+^*(-\omega, -\mathbf{k})$, which is consistent with the property $\varepsilon(-\omega) = \varepsilon^*(\omega)$. One finally obtains

$$\varepsilon(\omega, \mathbf{k}) = 1 + \frac{\omega_0^2 \chi_r}{\omega_0^2 + v^2 \mathbf{k}^2 - \omega^2 - \omega^2 \lambda_R(\omega)}, \tag{2.27}$$

where the function λ_R is given by

$$\lambda_R(\omega) = \frac{1}{\varrho^2} \int_0^\infty \mathrm{d}\omega' \frac{w(\omega')^2}{\omega'^2 - \omega^2 - \mathrm{i}\epsilon \,\mathrm{sign}(\omega)}. \tag{2.28}$$

Formula (2.27) is valid also in the whole $\mathbf{k}$ space. As a function of complex frequency, $\varepsilon(\omega, \mathbf{k})$ is analytic in the upper half-plane [5, 15], and is therefore consistent with the causality requirement and fulfils the Kramers–Kronig relations. The displacement field in the entire reciprocal space is given by

$$\mathbf{D}(\omega, \mathbf{k}) = \varepsilon_0 \mathbf{E}(\omega, \mathbf{k}) + \frac{\varepsilon_0 \omega_0^2 \chi_r}{\omega_0^2 + v^2 \mathbf{k}^2 - \omega^2 - \omega^2 \lambda_R(\omega)} \mathbf{E}(\omega, \mathbf{k}), \tag{2.29}$$

so that

$$\mathbf{D}(t, \mathbf{x}) = \varepsilon_0 \mathbf{E}(t, \mathbf{x}) + \int_{-\infty}^{\infty} \mathrm{d}t' \int \mathrm{d}^3 x' \, \Gamma_R(t - t', \mathbf{x} - \mathbf{x}') \mathbf{E}(t', \mathbf{x}'), \tag{2.30}$$

where the response function Γ_R in the configuration space has the form

$$\Gamma_R(t, \mathbf{x}) = \varepsilon_0 \omega_0^2 \chi_r \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{x}} \frac{1}{\omega_0^2 + v^2 \mathbf{k}^2 - \omega^2 - \omega^2 \lambda_R(\omega)}. \tag{2.31}$$

Since the denominator of the integrand is analytic in the upper half-plane of complex ω and has poles only in the lower half-plane, the response function Γ_R is zero for $t < 0$. The time integration in (2.30) extends therefore from $-\infty$ to the actual time t, in agreement with the causality requirements.

3. Full quantization of the model

In this section full path-integral quantization of the model described by the microscopic Lagrangian density (2.4) will be presented. For this purpose the Lagrangian density (2.4) will be amended by source terms and then the integration over all the fields, including also the electromagnetic field, will be performed. The resulting object is the generating functional of the propagators from which the propagators can be calculated by functional differentiation with respect to the source terms [26]. The issue of gauge invariance will also be discussed.

3.1. Generating functional

To perform the quantization in a complete and consistent way one should start from the Hamiltonian form of the path-integral expression for the transition amplitude [32, 34]. To construct the Hamiltonian one has to express the Lagrangian by electromagnetic potential. Together with the source terms this gives

$$\mathcal{L} = \mathcal{L}_{\text{em}} + \mathcal{L}_P + \mathcal{L}_Y + \mathcal{L}_{\text{int}} + \mathbf{A}\cdot\mathbf{J} + \mathbf{P}\cdot\mathbf{g}, \tag{3.1}$$

where

$$\mathcal{L}_{\text{em}} = \frac{\varepsilon_0}{2}(-\dot{\mathbf{A}} - \nabla\varphi)^2 - \frac{1}{2\mu_0}(\nabla\times\mathbf{A})^2, \tag{3.2a}$$

$$\mathcal{L}_P = \frac{1}{2\varepsilon_0\omega_0^2\chi_r}\left(\dot{\mathbf{P}}^2 - v^2\frac{\partial P_i}{\partial x_j}\frac{\partial P_i}{\partial x_j} - \omega_0^2\mathbf{P}^2\right), \tag{3.2b}$$

$$\mathcal{L}_Y = \int_0^\infty \mathrm{d}\omega'\left(\frac{\varrho}{2}\dot{\mathbf{Y}}^2_{\omega'} - \frac{\varrho}{2}\omega'^2\mathbf{Y}^2_{\omega'}\right), \tag{3.2c}$$

$$\mathcal{L}_{\text{int}} = \mathbf{P}\cdot(-\dot{\mathbf{A}} - \nabla\varphi) - \int_0^\infty \mathrm{d}\omega'\, f(\omega')\mathbf{P}\cdot\dot{\mathbf{Y}}_{\omega'}, \tag{3.2d}$$

where $\mathbf{J}$ and $\mathbf{g}$ are, respectively, sources of the electromagnetic and polarization field. To find the Hamiltonian it is necessary to determine canonical momenta. Since the time derivative of the scalar potential φ does not appear in the Lagrangian density, there is no momentum conjugate to φ. The canonical momenta can be determined provided the gauge has been fixed. The Weyl gauge $\varphi = 0$ will be chosen for the purpose of calculating the conjugate momenta,

$$\Pi_i^A = \frac{\partial\mathcal{L}}{\partial\dot{A}_i} = \varepsilon_0\dot{A}_i - P_i = -D_i, \tag{3.3a}$$

$$\Pi_i^P = \frac{\partial\mathcal{L}}{\partial\dot{P}_i} = \left(\varepsilon_0\omega_0^2\chi_r\right)^{-1}\dot{P}_i, \tag{3.3b}$$

$$\Pi_{\omega i}^Y = \frac{\partial\mathcal{L}}{\partial\dot{Y}_{\omega i}} = \varrho\dot{Y}_{\omega i} - f(\omega)P_i, \tag{3.3c}$$

where $\Pi_i^A\ldots$ denote momenta conjugate to the corresponding fields and D_i is the electric displacement field. The Hamiltonian density,

$$\mathcal{H} = \Pi_i^A\dot{A}_i + \Pi_i^P\dot{P}_i + \int_0^\infty \mathrm{d}\omega\,\Pi_{\omega i}^Y\dot{Y}_{\omega i} - \mathcal{L}, \tag{3.4}$$

is gauge invariant.The velocities in (3.4) have to be expressed by fields and conjugate momenta.

Electrodynamics is a constrained theory, i.e. apart from the dynamical equations containing time derivatives of the fields, there is also a constraint equation in the form of Gauss law,

$$\nabla\cdot\mathbf{D} - \eta = 0, \tag{3.5}$$

where η is the external charge density coupling to the scalar potential. Moreover, to perform the path-integral quantization correctly it is necessary to fix the gauge by a condition of the type $\gamma_g(\mathbf{A}) = 0$ [34]. Taking into account the constraint (3.5) and the gauge condition one obtains the expression for the generating functional in the form [34]

$$C[\mathbf{J},\mathbf{g},\eta] = \mathcal{N}\int[\mathrm{d}\mathbf{A}][\mathrm{d}\mathbf{P}][\mathrm{d}\mathbf{Y}][\mathrm{d}\mathbf{D}][\mathrm{d}\Pi^P][\mathrm{d}\Pi^Y]\delta(\nabla\cdot\mathbf{D}-\eta)\delta(\gamma_g(\mathbf{A}))$$
$$\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_x\left(-D_i\dot{A}_i + \Pi_i^P\dot{P}_i + \int_0^\infty \mathrm{d}\omega\,\Pi_{\omega i}^Y\cdot\dot{Y}_{\omega i} - \mathcal{H}\right)\right], \tag{3.6}$$

where $\mathcal{N}$ is the normalization factor. In general, the determinant of the Poisson bracket of the constraint and gauge condition, $\det\{\nabla\cdot\mathbf{D}-\eta,\gamma_g(\mathbf{A})\}$, should be included as a factor [34], but in the Abelian gauge theory it is independent of the fields and can be absorbed in the normalization $\mathcal{N}$. The constraint functional delta is further written in the form of 'functional Fourier representation' [34],

$$\delta(\nabla\cdot\mathbf{D}-\eta)\propto\int[d\varphi]\exp\left[\frac{\mathrm{i}}{\hbar}\int_x\varphi(\nabla\cdot\mathbf{D}-\eta)\right], \tag{3.7}$$

with φ identified as the scalar potential. After substituting (3.7) into (3.6) one can perform functional integrations over momenta. Since the Hamiltonian is quadratic in momenta one gets, after performing the Gaussian integrals,

$$C[\mathbf{J},\mathbf{g},\eta]=\mathcal{N}\int[\mathrm{d}\mathbf{A}][\mathrm{d}\varphi][\mathrm{d}\mathbf{P}][\mathrm{d}\mathbf{Y}]\delta(\gamma_g(\mathbf{A}))\exp\left[\frac{\mathrm{i}}{\hbar}\int_x(\mathcal{L}-\eta\varphi)\right], \tag{3.8}$$

where $\mathcal{L}$ is the Lagrangian density (3.1) and the 'Fourier representation' of the constraint delta resulted in a source term for the scalar potential. With the choice of the Coulomb gauge, $\nabla\cdot\mathbf{A}=0$, the gauge fixing delta is represented by

$$\delta(\nabla\cdot\mathbf{A})\propto\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{1}{2\mu_0\alpha}\int_x(\nabla\cdot\mathbf{A})^2\right], \tag{3.9}$$

with $\alpha\to 0$ at the end of calculations. Inserting (3.9) into (3.8) one obtains the functional integral for C with the Lagrangian density amended by the gauge fixing term

$$C[\mathbf{J},\mathbf{g},\eta]=\mathcal{N}\int[\mathrm{d}\mathbf{A}][\mathrm{d}\varphi][\mathrm{d}\mathbf{P}][\mathrm{d}\mathbf{Y}]\exp\left[\frac{\mathrm{i}}{\hbar}\int_x\left(\mathcal{L}-\frac{\mathrm{i}}{\hbar}\frac{1}{2\mu_0\alpha}\int_x(\nabla\cdot\mathbf{A})^2-\eta\varphi\right)\right]. \tag{3.10}$$

The normalization factor has to be determined from the condition $C[\mathbf{0},\mathbf{0}]=1$. For a quadratic action of a linear model, the normalization factor is equal to unity. The time-ordered propagator of the electromagnetic field is given by

$$D_{ij}(t-t',\mathbf{r}-\mathbf{r}')=\frac{\mathrm{i}}{\hbar}\langle T[A_i(t,\mathbf{r})A_j(t,\mathbf{r}')]\rangle=-\mathrm{i}\hbar\frac{\delta^2C[\mathbf{J},\mathbf{G}]}{\delta J_i(t,\mathbf{r})\delta J_j(t',\mathbf{r}')}\bigg|_{\mathbf{J}=0,\mathbf{g}=0,\eta=0}, \tag{3.11}$$

where $\langle\cdots\rangle$ denotes the ground state expectation value and T denotes the time-ordered product of the operators. In the present case

$$T[A_i(t,\mathbf{r})A_j(t,\mathbf{r}')]=\theta(t-t')A_i(t,\mathbf{r})A_j(t,\mathbf{r}')+\theta(t'-t)A_j(t',\mathbf{r}')A_i(t,\mathbf{r}). \tag{3.12}$$

The polarization field propagator is calculated in a similar way:

$$\Pi_{ij}(t-t',\mathbf{r}-\mathbf{r}')=\frac{\mathrm{i}}{\hbar}\langle T[P_i(t,\mathbf{r})P_j(t,\mathbf{r}')]\rangle=-\mathrm{i}\hbar\frac{\delta^2C[\mathbf{J},\mathbf{G}]}{\delta g_i(t,\mathbf{r})\delta g_j(t',\mathbf{r}')}\bigg|_{\mathbf{J}=0,\mathbf{g}=0,\eta=0}, \tag{3.13}$$

and also the propagator of scalar potential

$$S(t-t',\mathbf{r}-\mathbf{r}')=\frac{\mathrm{i}}{\hbar}\langle T[\varphi(t,\mathbf{r})\varphi(t,\mathbf{r}')]\rangle=-\mathrm{i}\hbar\frac{\delta^2C[\mathbf{J},\mathbf{G}]}{\delta\eta(t,\mathbf{r})\delta\eta(t',\mathbf{r}')}\bigg|_{\mathbf{J}=0,\mathbf{g}=0,\eta=0}. \tag{3.14}$$

Functional integration over the reservoir field $\mathbf{Y}$ will be performed first. Since $\mathbf{Y}$ does not couple to the electromagnetic field, the integration can be done in the same way as in section 2. The result is

$$\begin{aligned}C[\mathbf{J},\mathbf{g},\eta]=\mathcal{N}\int[\mathrm{d}\mathbf{A}][\mathrm{d}\varphi][\mathrm{d}\mathbf{P}]\exp\left\{\frac{\mathrm{i}}{\hbar}\int_x[\mathcal{L}_{\mathrm{em}}+\mathcal{L}_P+\mathbf{P}\cdot(-\dot{\mathbf{A}}-\nabla\varphi)]+\frac{\mathrm{i}}{\hbar}S_P\right\}\\ \times\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{1}{2\mu_0\alpha}\int_x(\nabla\cdot\mathbf{A})^2+\frac{\mathrm{i}}{\hbar}\int_x(\mathbf{J}\cdot\mathbf{A}+\mathbf{P}\cdot\mathbf{g}-\eta\varphi)\right],\end{aligned} \tag{3.15}$$

where

$$S_P = \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}t' \int \mathrm{d}^3x \int_0^\infty \mathrm{d}\omega' \, \frac{1}{\varrho} f(\omega')^2 \dot{\mathbf{P}}(t,\mathbf{r}) D_F(t-t',\omega') \dot{\mathbf{P}}(t',\mathbf{r}). \tag{3.16}$$

There are still three functional integrals to perform, all of them Gaussian. The calculation is straightforward though quite lengthy with increasing degree of complexity when consecutive integrations are done. To avoid technical details as much as possible, the form of generating functional after the integrations over vector and scalar potentials is given below,

$$\begin{aligned} C[\mathbf{J},\mathbf{g},\eta] = \int [\mathrm{d}\mathbf{P}] \exp\left[\frac{\mathrm{i}}{\hbar}\left(S_P + \int_x \mathcal{L}_P\right) + \frac{\mathrm{i}}{\hbar}\int_x \mathbf{P}\cdot\mathbf{g}\right] \\ \times \exp\left\{\frac{\mathrm{i}}{2\hbar}\int_x\int_{x'} [\dot{P}_i(x) + J_i(x)] D_{ij}^{(0)}(x-x')[\dot{P}_i(x') + J_i(x')]\right\} \\ \times \exp\left[\frac{\mathrm{i}}{2\hbar}\int_x\int_{x'} \sigma(x) S^{(0)}(x-x')\sigma(x')\right], \end{aligned} \tag{3.17}$$

where the source function $\sigma(x)$ has the form

$$\sigma(x) = \eta(x) - \partial_i P_i + \int_{x'} \varepsilon_0 \partial_t \partial_i D_{ij}^{(0)}(x-x')[\dot{P}_j(x') + J_j(x')], \tag{3.18}$$

and the free propagators of vector and scalar potential, $D_{ij}^{(0)}$ and $S^{(0)}$, respectively, have the Fourier transforms

$$\widetilde{D}_{ij}^{(0)}(\omega,\mathbf{k}) = -\frac{1}{\varepsilon_0}\frac{1}{\omega^2 - c^2\mathbf{k}^2 + \mathrm{i}\epsilon}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) - \frac{1}{\varepsilon_0\omega^2 - \mathbf{k}^2/(\alpha\mu_0)}\frac{k_i k_j}{\mathbf{k}^2}, \tag{3.19a}$$

$$\widetilde{S}^{(0)}(\omega,\mathbf{k}) = -\frac{1}{\varepsilon_0\mathbf{k}^2}\left(1 - \alpha\frac{\omega^2}{c^2\mathbf{k}^2}\right). \tag{3.19b}$$

Both propagators of the electromagnetic field are gauge dependent, as can be seen from the presence of the parameter α. The transverse part of the vector potential propagator is gauge invariant, so it does not depend on α, and in the Coulomb gauge limit $\alpha \to 0$ the longitudinal part vanishes. The propagator of scalar potential corresponds in this limit to the instantaneous Coulomb interaction.

To determine the polarization field propagator and perform the remaining integration in (3.17) one has to collect terms quadratic in $\mathbf{P}$. The resulting equation for the Fourier transform of the polarization field propagator $\widetilde{\Pi}_{ij}$ reads

$$\left[\left(\varepsilon_0\omega_0^2\chi_r\right)^{-1}\left(\omega_0^2 + v^2\mathbf{k}^2 - \omega^2 - \omega^2\lambda_F - \omega^2\widetilde{D}_T^{(0)}\right)\delta_{ij} + \left(\omega^2\widetilde{D}_T^{(0)} + \varepsilon_0^{-1}\right)(k_i k_j/\mathbf{k}^2)\right]\widetilde{\Pi}_{jl} = \delta_{jl}, \tag{3.20}$$

where $\widetilde{D}_T^{(0)}$ denotes the transverse part of the vector field propagator

$$\widetilde{D}_T^{(0)} = -\frac{1}{\varepsilon_0}\frac{1}{\omega^2 - c^2\mathbf{k}^2 + \mathrm{i}\epsilon} \tag{3.21}$$

and λ_F is given by (2.18). The terms dependent on the parameter α cancelled in (3.20) as expected, since the polarization field is gauge independent, and so is its propagator. Equation (3.20) can be easily solved,

$$\widetilde{\Pi}_{ij}(\omega,\mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right)\frac{\omega^2 - c^2\mathbf{k}^2}{\omega^2\widetilde{\varepsilon}(\omega,\mathbf{k}) - c^2\mathbf{k}^2}\widetilde{\Gamma}(\omega,\mathbf{k}) + \frac{k_i k_j}{\mathbf{k}^2}\frac{\widetilde{\Gamma}(\omega,\mathbf{k})}{\widetilde{\varepsilon}(\omega,\mathbf{k})}, \tag{3.22}$$

where $\widetilde{\Gamma}$ is the Fourier transform (2.19) of the integral kernel Γ, and

$$\widetilde{\varepsilon}(\omega,\mathbf{k}) = 1 + \varepsilon_0^{-1}\widetilde{\Gamma}(\omega,\mathbf{k}). \tag{3.23}$$

The final form of the generating functional reads

$$C[\mathbf{J},\mathbf{g},\eta] = \exp\left[\frac{\mathrm{i}}{2\hbar}\int_{x_1}\int_{x_2} J_i(x_1)D_{ij}^{(0)}(x_1-x_2)J_j(x_2)\right] \times \exp\left[\frac{\mathrm{i}}{2\hbar}\int_{x_1}\int_{x_2}\tau(x_1)S^{(0)}(x_1-x_2)\tau(x_2)\right] \times \exp\left\{\frac{\mathrm{i}}{2\hbar}\int_{x_1}\int_{x_2}[g_i(x_1)-\mathcal{J}_i(x_1)]\Pi_{ij}(x_1-x_2)[g_j(x_2)-\mathcal{J}_j(x_2)]\right\}, \quad (3.24)$$

where the sources τ and $\mathcal{J}_i$ are given by

$$\tau(x) = \eta(x) + \varepsilon_0\int_{x'}\partial_t\partial_i D_{ij}^{(0)}J_j(x'), \quad (3.25a)$$

$$\mathcal{J}_i(x) = \int_{x'}\partial_t D_{Tij}^{(0)}(x-x')J_j(x') + \int_{x'}\int_{x''}\Delta_i(x-x')S^{(0)}(x'-x'')\eta(x''), \quad (3.25b)$$

with

$$\Delta_i(x) = -\partial_i\delta(x) + \varepsilon_0\partial_t^2\partial_j D_{ij}^{(0)}(x). \quad (3.26)$$

3.2. Interacting propagators

The generating functional calculated in the previous section will now be used to calculate the propagators of the fields including the effects of interaction between matter and the electromagnetic field. Using (3.13) and (3.24), one easily notes that the full propagator of the polarization field has the form (3.22). For the full propagator of the vector potential, one obtains from (3.11) and (3.24)

$$\widetilde{D}_{ij}(\omega,\mathbf{k}) = -\frac{1}{\varepsilon_0}\frac{1}{\omega^2\widetilde{\varepsilon}(\omega,\mathbf{k})-c^2\mathbf{k}^2}\left(\delta_{ij}-\frac{k_ik_j}{\mathbf{k}^2}\right)+\frac{k_ik_j}{\mathbf{k}^2}\frac{\alpha\mu_0}{\mathbf{k}^2}, \quad (3.27)$$

whereas for the scalar potential propagator one gets

$$S(\omega,\mathbf{k}) = -\frac{1}{\varepsilon_0\mathbf{k}^2\widetilde{\varepsilon}(\omega,\mathbf{k})}+\alpha\frac{\omega^2}{\varepsilon_0c^2k^4}. \quad (3.28)$$

As expected, the transverse part of the vector potential propagator is independent of the parameter α, and in the $\alpha\to 0$ limit the propagator is purely transverse. The scalar potential propagator in the $\alpha\to 0$ limit corresponds for $\omega = 0$ (static limit) to the screened Coulomb interaction, the screening being due to the $\mathbf{k}$ dependence of $\widetilde{\varepsilon}$. For real positive frequencies, $\widetilde{\varepsilon}$ is equal to the dielectric function, and $\widetilde{\varepsilon} = \varepsilon^*$ for $\omega < 0$. Zeros of the denominator of the vector potential propagator determine the dispersion relation between frequency and wave vector for the polariton modes [16],

$$\omega^2\widetilde{\varepsilon}(\omega,\mathbf{k}) - c^2\mathbf{k}^2 = 0. \quad (3.29)$$

Since the dielectric function $\varepsilon(\omega,\mathbf{k})$ is complex for an absorbing medium, this relation cannot be fulfilled for real frequencies and wave vectors. For real wave vector one gets solutions with complex frequencies, which for $\mathrm{Re}\,\omega > 0$ have the negative imaginary part, and for the negative real part the imaginary part is positive, corresponding to the Feynman-type structure of singularities of the propagator.

4. Equal time commutators

Basic objects in the path-integral quantization of fields are the propagators. For various applications the key objects are field operators, which, among others, should fulfil correct canonical commutation rules. In the case of the vector potential, for which the conjugate momentum is $-D_i$, one would expect in the Coulomb gauge

$$[A_i(t,\mathbf{r}),\, D_j(t,\mathbf{r}')] = -\mathrm{i}\hbar(\delta_{ij} - \Delta^{-1}\partial_i\partial_j)\delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{4.1}$$

where Δ denotes, here, the Laplace operator. It is not however obvious that for interacting quantum fields such a 'naive' equal time commutation relation (ECTR) is fulfilled [35]. The point is that additional terms may appear on the right-hand side, for instance involving spatial derivatives of delta function (Schwinger terms). One has therefore to check the ECTR using more sophisticated tools, usually referred to as BJL (Bjorken, Johnson, Low) limit [27, 28]. The ECTR will be checked here with the direct use of the propagators derived in the previous section using a scheme described in Bjorken's paper [28]

It is sufficient to consider the ground state expectation values of the ECTR. This will be done by constructing the quantity

$$C_{ij}(\omega,\mathbf{k}) = \frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{r})}\langle T[A_i(t,\mathbf{r})D_j(0)]\rangle. \tag{4.2}$$

Using further definition of the time-ordered product and the formulae

$$\mathrm{e}^{\mathrm{i}\omega t}\theta(\pm t) = \frac{1}{2\pi\,\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\pm\mathrm{i}\omega' t}}{\omega'\mp\omega-\mathrm{i}\epsilon}\,\mathrm{d}\omega', \tag{4.3}$$

one can write (4.2) as

$$C_{ij}(\omega,\mathbf{k}) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}\omega'\left[\frac{c_{ij}(\omega',\mathbf{k})}{\omega'-\omega-\mathrm{i}\epsilon}+\frac{\widetilde{c}_{ji}(\omega',-\mathbf{k})}{\omega'+\omega-\mathrm{i}\epsilon}\right], \tag{4.4}$$

where

$$c_{ij}(\omega,\mathbf{k}) = \int \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{r})}\langle A_i(t,\mathbf{r})D_j(0)\rangle, \tag{4.5a}$$

$$\widetilde{c}_{ji}(\omega,\mathbf{k}) = \int \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{r})}\langle D_j(0)A_i(t,\mathbf{r})\rangle. \tag{4.5b}$$

On the other hand, as follows from (4.5*a*) and (4.5*b*)

$$\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\langle[A_i(0,\mathbf{r}),\,D_j(0)]\rangle = \int_{-\infty}^{\infty}\frac{\mathrm{d}\omega'}{2\pi}[c_{ij}(\omega',\mathbf{k})-\widetilde{c}_{ji}(\omega',-\mathbf{k})]. \tag{4.6}$$

It follows then from (4.4) and (4.6) that for $\omega\to\infty$

$$C_{ij}(\omega,\mathbf{k}) \to -\frac{1}{\hbar\omega}\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\langle[A_i(0,\mathbf{r}),\,D_j(0)]\rangle, \tag{4.7}$$

which shows that the ground state expectation value of the ECTR can be derived from the time-ordered products of the operators. Ground state expectation values of time-ordered products are the propagators of the fields; it follows, therefore, that the ECTR can be derived with the use of propagators. However, it is not *a priori* guaranteed that using the field operators and the propagators calculated in an independent way (e.g. by path-integral method) one would obtain the same result. The difference (which of course may be zero) contributes to the Schwinger terms in the ECTR.

Let $\widetilde{C}_{ij}$ denote the quantity corresponding to C_{ij} but calculated with the direct use of the propagators found in the previous section. The ECTR will be derived by calculating the $\omega \to \infty$ limit of $\widetilde{C}_{ij}$. The electric displacement field is given as

$$D_i(t,\mathbf{r}) = -\varepsilon_0[\dot{A}_i(t,\mathbf{r}) + \partial_i\varphi(t,\mathbf{r})] + P_i(t,\mathbf{r}). \tag{4.8}$$

Thus

$$\begin{aligned}\langle T[A_i(t,\mathbf{r})D_j(0)]\rangle &= \varepsilon_0\partial_t\langle T[A_i(t,\mathbf{r})A_j(0)]\rangle - \varepsilon_0\delta(t)\langle[A_i(0,\mathbf{r}),A_j(0)]\rangle \\ &\quad + \varepsilon_0\partial_j\langle T[A_i(t,\mathbf{r})\varphi(0)]\rangle + \langle T[A_i(t,\mathbf{r})P_j(0)]\rangle.\end{aligned} \tag{4.9}$$

The 'counterpart' $\widetilde{C}_{ij}$ of C_{ij} can therefore be expressed as

$$\begin{aligned}\widetilde{C}_{ij}(\omega,\mathbf{k}) &= \varepsilon_0\int \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{r})}\partial_t D_{ij}(t,\mathbf{r}) - \varepsilon_0\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\langle[A_i(0,\mathbf{r}),A_j(0)]\rangle \\ &\quad + \varepsilon_0\int \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{r})}\partial_j\Phi_i(t,\mathbf{r}) + \varepsilon_0\int \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{r})}K_{ij}(t,\mathbf{r}),\end{aligned} \tag{4.10}$$

where D_{ij} is the vector potential propagator with the Fourier transform (3.27), and the two remaining propagators are

$$\Phi_i(t,\mathbf{r}) = -\mathrm{i}\hbar\frac{\delta^2 C}{\delta J_i(t,\mathbf{r})\delta\eta(0)}, \tag{4.11a}$$

$$K_{ij}(t,\mathbf{r}) = -\mathrm{i}\hbar\frac{\delta^2 C}{\delta J_i(t,\mathbf{r})\delta g_j(0)}. \tag{4.11b}$$

The second term in (4.10) can be calculated by examining the $\omega \to \infty$ limit of the vector potential propagator, according to the general rule expressed in (4.7):

$$\begin{aligned}\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\langle[A_i(0,\mathbf{r}),A_j(0)]\rangle &= -\hbar\omega\int \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{r})}D_{ij}(t,\mathbf{r})\Big|_{\omega\to\infty} \\ &= -\hbar\omega\widetilde{D}_{ij}(\omega,\mathbf{k})|_{\omega\to\infty}.\end{aligned} \tag{4.12}$$

The large frequency behaviour can be obtained from (3.27) taking into account that in this limit $\widetilde{\varepsilon} \to 1$. Therefore, the transverse part goes to zero like ω^{-2}, and the longitudinal part vanishes in the Coulomb gauge limit $\alpha = 0$. This proves that the equal time commutator of vector potentials vanishes, as one would expect.

Explicit calculation using (3.24) shows that Φ_i is proportional to the longitudinal part of the vector potential propagator and vanishes for $\alpha = 0$. Further

$$\int \mathrm{d}t\,\mathrm{d}^3x\,\mathrm{e}^{\mathrm{i}(\omega t-\mathbf{k}\cdot\mathbf{r})}K_{ij}(t,\mathbf{r}) = \widetilde{K}_{ij}(\omega,\mathbf{k}) = -\mathrm{i}\omega\left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right)\frac{(\omega^2 - c^2\mathbf{k}^2)\widetilde{\Gamma}}{\varepsilon_0(\omega^2\widetilde{\varepsilon} - c^2\mathbf{k}^2)^2} \tag{4.13}$$

vanishes like ω^{-3}, and does not therefore contribute to the ECTR.

The term which still has to be examined is the first contribution to the right-hand side of (4.10). It is given by $-\mathrm{i}\omega\varepsilon_0\widetilde{D}_{ij}(\omega,\mathbf{k})$, and falls off to zero as ω^{-1}. Explicitly

$$\widetilde{C}_{ij}(\omega,\mathbf{k}) \to \frac{\mathrm{i}}{\omega}\left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right). \tag{4.14}$$

This type of behaviour allows us to compare (4.14) with (4.7), which leads to the relation

$$-\frac{1}{\hbar}\int \mathrm{d}^3x\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\langle[A_i(0,\mathbf{r}),D_j(0)]\rangle = \mathrm{i}\left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right), \tag{4.15}$$

which is consistent with the 'naive' ECTR (4.1).

5. Field operators

5.1. Construction of field operators

Path-integral quantization of a field theory model consists basically in construction of the propagators. The objects which are equally important for applications of quantum electrodynamics of dielectric media are the field operators which in this quantization scheme are not the primary quantities and have to be retrieved from the propagators. Due to the lack of the dispersion relation in a dispersive medium, plane-wave decompositions of field operators must contain independent integrations over frequency and wave vector. For the purpose of construction of field operators two sets of annihilation and creation operators are introduced, $C_T^{(\lambda)}(\omega, \mathbf{k})$ and $C_L(\omega, \mathbf{k})$, and their Hermitian conjugates. The subscript T/L corresponds to transverse/longitudinal effective excitations in the medium (polariton modes). The commutation relations read

$$\left[C_T^{(\lambda)}(\omega, \mathbf{k}), C_T^{(\lambda')\dagger}(\omega', \mathbf{k}')\right] = (2\pi)^4 \delta_{\lambda\lambda'} \delta(\omega - \omega') \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{5.1a}$$

$$\left[C_L(\omega, \mathbf{k}), C_L^{\dagger}(\omega', \mathbf{k}')\right] = (2\pi)^4 \delta(\omega - \omega') \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \tag{5.1b}$$

with the remaining commutators vanishing. The transverse operator of the vector potential in the Coulomb gauge has the plane-wave decomposition

$$\mathbf{A}(t, \mathbf{r}) = \sum_{\lambda} \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[\phi(\omega, \mathbf{k}) \mathbf{e}^{(\lambda)}(\mathbf{k}) C_T^{(\lambda)}(\omega, \mathbf{k}) \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{r}} + \mathrm{h.c.}\right], \tag{5.2}$$

where h.c. denotes the Hermitian conjugate and $e^{(\lambda)}(\mathbf{k})$ is the polarization vector orthogonal to $\mathbf{k}$ and fulfilling the summation formula

$$\sum_{\lambda} e_i^{(\lambda)*}(\mathbf{k}) e_j^{(\lambda)}(\mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right). \tag{5.3}$$

The complex-valued function ϕ will be determined in a similar way as in [36] or [5] by calculating first the propagator as the expectation value of the time-ordered product of field operators and then comparing the resulting expression with the propagator obtained in section 3.2. This procedure leads to the equation

$$\frac{1}{2\pi\hbar} \int_0^{\infty} \mathrm{d}\omega' |\phi(\omega', \mathbf{k})|^2 \left(\frac{1}{\omega - \omega' + \mathrm{i}\epsilon} - \frac{1}{\omega + \omega' - \mathrm{i}\epsilon}\right) = \frac{1}{\varepsilon_0} \frac{1}{\omega^2 \widetilde{\varepsilon}(\omega, \mathbf{k}) - c^2\mathbf{k}^2}. \tag{5.4}$$

It is shown in appendix A that equation (5.4) has solution in the form

$$\phi(\omega, \mathbf{k}) = \left(\frac{2\hbar}{\varepsilon_0}\right)^{1/2} \frac{\omega\{\mathrm{Im}[\varepsilon(\omega, \mathbf{k})]\}^{1/2}}{\omega^2 \varepsilon(\omega, \mathbf{k}) - c^2\mathbf{k}^2}. \tag{5.5}$$

Operator of the polarization field can be determined in a similar way. Assuming the plane-wave decomposition

$$\begin{aligned}\mathbf{P}(t, \mathbf{r}) = &\sum_{\lambda} \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[p(\omega, \mathbf{k}) \mathbf{e}^{(\lambda)}(\mathbf{k}) C_T^{(\lambda)}(\omega, \mathbf{k})\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{r}} + h.c\right] \\ &+ \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[q(\omega, \mathbf{k}) \frac{\mathbf{k}}{|\mathbf{k}|} C_L(\omega, \mathbf{k})\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\mathbf{k}\cdot\mathbf{r}} + h.c\right],\end{aligned} \tag{5.6}$$

calculating the propagator as the expectation value of the time-ordered product and comparing with expression (3.22), one obtains equations for the p and q functions

$$\frac{1}{2\pi\hbar}\int_0^\infty \mathrm{d}\omega' |p(\omega',\mathbf{k})|^2 \left(\frac{1}{\omega-\omega'+\mathrm{i}\epsilon}-\frac{1}{\omega+\omega'-\mathrm{i}\epsilon}\right) = -\widetilde{\Gamma}(\omega,\mathbf{k})\frac{\omega^2-c^2\mathbf{k}^2}{\omega^2\widetilde{\varepsilon}(\omega,\mathbf{k})-c^2\mathbf{k}^2}, \tag{5.7a}$$

$$\frac{1}{2\pi\hbar}\int_0^\infty \mathrm{d}\omega' |q(\omega',\mathbf{k})|^2 \left(\frac{1}{\omega-\omega'+\mathrm{i}\epsilon}-\frac{1}{\omega+\omega'-\mathrm{i}\epsilon}\right) = -\frac{\widetilde{\Gamma}(\omega,\mathbf{k})}{\widetilde{\varepsilon}(\omega,\mathbf{k})}, \tag{5.7b}$$

which can be solved in the same way as the previous equation for ϕ. It is shown in appendix A that

$$p(\omega,\mathbf{k}) = -\mathrm{i}\varepsilon_0\left(\frac{2\hbar}{\varepsilon_0}\right)^{1/2}\frac{(\omega^2-c^2\mathbf{k}^2)\{\mathrm{Im}[\varepsilon(\omega,\mathbf{k})]\}^{1/2}}{\omega^2\varepsilon(\omega,\mathbf{k})-c^2\mathbf{k}^2} = -\mathrm{i}\varepsilon_0\frac{\omega^2-c^2\mathbf{k}^2}{\omega}\phi(\omega,\mathbf{k}), \tag{5.8a}$$

$$q(\omega,\mathbf{k}) = -\mathrm{i}\varepsilon_0\left(\frac{2\hbar}{\varepsilon_0}\right)^{1/2}\frac{\{\mathrm{Im}[\varepsilon(\omega,\mathbf{k})]\}^{1/2}}{\varepsilon(\omega,\mathbf{k})}. \tag{5.8b}$$

Plane-wave decomposition of the scalar potential contains only longitudinal modes, and proceeding in the same way as for $\mathbf{A}$ and $\mathbf{P}$ one obtains

$$\varphi(t,\mathbf{r}) = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{|\mathbf{k}|}\left[-\frac{\mathrm{i}}{\varepsilon_0}q(\omega,\mathbf{k})C_L(\omega,\mathbf{k})\,\mathrm{e}^{-\mathrm{i}\omega t+\mathrm{i}\mathbf{k}\cdot\mathbf{r}}+h.c\right]. \tag{5.9}$$

Next step is the construction of gauge-independent operators of the electric field, $\mathbf{E} = -\dot{\mathbf{A}} - \nabla\varphi$, and of the displacement field, $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$. Potentials and the polarization field can be decomposed into positive and negative frequency parts. Denoting the positive frequency contribution by the superscript '(+)' one has in the reciprocal space

$$\mathbf{A}^{(+)}(\omega,\mathbf{k}) = \phi(\omega,\mathbf{k})\sum_\lambda \mathbf{e}^{(\lambda)}(\mathbf{k})C_T^{(\lambda)}(\omega,\mathbf{k}), \tag{5.10a}$$

$$\mathbf{P}^{(+)}(\omega,\mathbf{k}) = p(\omega,\mathbf{k})\sum_\lambda \mathbf{e}^{(\lambda)}(\mathbf{k})C_T^{(\lambda)}(\omega,\mathbf{k}) + \frac{\mathbf{k}}{|\mathbf{k}|}q(\omega,\mathbf{k})C_L(\omega,\mathbf{k}), \tag{5.10b}$$

$$\varphi^{(+)}(\omega,\mathbf{k}) = -\frac{\mathrm{i}}{\varepsilon_0}q(\omega,\mathbf{k})C_L(\omega,\mathbf{k}). \tag{5.10c}$$

For the operator of the electric field one gets

$$\mathbf{E}^{(+)}(\omega,\mathbf{k}) = \mathrm{i}\omega\phi(\omega,\mathbf{k})\sum_\lambda \mathbf{e}^{(\lambda)}(\mathbf{k})C_T^{(\lambda)}(\omega,\mathbf{k}) - \frac{1}{\varepsilon_0}\frac{\mathbf{k}}{|\mathbf{k}|}q(\omega,\mathbf{k})C_L(\omega,\mathbf{k}). \tag{5.11}$$

The electric field has in general transverse and longitudinal parts, with the latter vanishing for the non-absorptive medium, for which $\mathrm{Im}[\varepsilon(\omega,\mathbf{k})] = 0$. Note that the longitudinal part does not vanish for a homogeneous absorbing medium, also when the wave vector dispersion (spatial non-locality of constitutive equations) is disregarded. Calculating the displacement field one finds

$$\mathbf{D}^{(+)}(\omega,\mathbf{k}) = \mathrm{i}\varepsilon_0\phi(\omega,\mathbf{k})\frac{c^2\mathbf{k}^2}{\omega}\sum_\lambda \mathbf{e}^{(\lambda)}(\mathbf{k})C_T^{(\lambda)}(\omega,\mathbf{k}), \tag{5.12}$$

which is purely transverse. For completeness, the positive frequency part of the magnetic induction field is also given:

$$\mathbf{B}^{(+)}(\omega,\mathbf{k}) = \mathrm{i}\phi(\omega,\mathbf{k})\sum_\lambda \mathbf{k}\times\mathbf{e}^{(\lambda)}(\mathbf{k})C_T^{(\lambda)}(\omega,\mathbf{k}). \tag{5.13}$$

5.2. Quantum constitutive equation

As long as the electromagnetic field in an absorptive dielectric medium is not quantized the constitutive equation has the homogeneous form (2.30). In the fully quantized theory the electric displacement field $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ was obtained using the electric and polarization fields calculated as independent objects, without assuming any relation between them. It is well known that a direct consequence of the presence of absorption is that the constitutive equation for the quantized fields does no longer have the form (2.30), but contains an additional noise term due to absorption [15, 37], so that for positive frequency part of the displacement field one has in the reciprocal space

$$\mathbf{D}^{(+)}(\omega, \mathbf{k}) = \varepsilon_0\varepsilon(\omega, \mathbf{k})\mathbf{E}^{(+)}(\omega, \mathbf{k}) + \mathbf{F}^{(+)}(\omega, \mathbf{k}), \tag{5.14}$$

where the noise field has been denoted by $\mathbf{F}$. Using (5.11) and (5.12) one finds

$$\mathbf{F}^{(+)}(\omega, \mathbf{k}) = -\mathrm{i}\varepsilon_0 \left(\frac{2\hbar}{\varepsilon_0}\right)^{1/2} \{\mathrm{Im}[\varepsilon(\omega, \mathbf{k})]\}^{1/2} \left[\sum_{\lambda} \mathbf{e}^{(\lambda)}(\mathbf{k}) C_T^{(\lambda)}(\omega, \mathbf{k}) + \frac{\mathbf{k}}{|\mathbf{k}|} C_L(\omega, \mathbf{k})\right], \tag{5.15}$$

which shows that the noise term vanishes in the non-absorbing medium for which the imaginary part of the dielectric function is equal to zero.

One can infer about a necessity to introduce an additional source term, corresponding to noise contribution on quantum level, also on the basis of purely classical considerations. The point is that with a complex dielectric function no non-trivial solutions of the homogeneous Maxwell equations exist if the frequency and wave vector in the reciprocal space are required to be real quantities. From homogeneous Maxwell equations one obtains the wave equation in the form

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -\mu_0 \frac{\partial^2 \mathbf{D}}{\partial t^2}. \tag{5.16}$$

Assuming that the classical constitutive equation $\mathbf{D}(\omega, \mathbf{k}) = \varepsilon_0\varepsilon(\omega, \mathbf{k})\mathbf{E}(\omega, \mathbf{k})$ in the reciprocal space one gets

$$\left[\mathbf{k}^2\delta_{ij} - \frac{\omega^2}{c^2}\varepsilon(\omega, \mathbf{k})\delta_{ij} - k_i k_j\right] E_j(\omega, \mathbf{k}) = 0. \tag{5.17}$$

For the complex dielectric function the dispersion equation

$$\det\left[\mathbf{k}^2\delta_{ij} - \frac{\omega^2}{c^2}\varepsilon(\omega, \mathbf{k})\delta_{ij} - k_i k_j\right] = 0 \tag{5.18}$$

cannot be fulfilled for real frequencies and wave vectors and, as a consequence, no non-vanishing solutions for the electric field could be found. With the lack of dispersion relation between real frequency and wave vector non-trivial solutions of the field equations can exist only in the presence of an additional current on the right-hand side of (5.17). Maxwell equations for the effective quantum fields $\mathbf{B}$ and $\mathbf{E}$ have the form

$$\varepsilon_0 \nabla \cdot (\mathbf{E} + \hat{\chi}\mathbf{E}) = -\nabla \cdot \mathbf{F}, \tag{5.19a}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5.19b}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{5.19c}$$

$$\nabla \times \mathbf{B} - \mu_0 \frac{\partial}{\partial t}(\mathbf{E} + \hat{\chi}\mathbf{E}) = \mu_0 \frac{\partial \mathbf{F}}{\partial t}, \tag{5.19d}$$

where **F** is the noise field and the shorthand notation has been used

$$\hat{\chi}\mathbf{E}(t,\mathbf{r}) = \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}^3\mathbf{r}'\, \chi(t-t', \mathbf{r}-\mathbf{r}')\mathbf{E}(t',\mathbf{r}'). \tag{5.20}$$

$\chi(t,\mathbf{r})$ is the response function $\chi(t,\mathbf{r}) = \varepsilon_0^{-1}\Gamma_R(t,\mathbf{r})$ (cf (2.31)). The noise charge density $\rho_n = -\nabla\cdot\mathbf{F}$ and the noise current density $\mathbf{j}_n = \partial_t\mathbf{F}$ [14] can be considered as sources of the **E** and **B** fields, though, strictly speaking, there are no 'true' external sources of the fields.

6. Final remarks

The path-integration method has been applied to the quantization of the electromagnetic field in dispersive dielectric media including also the wave vector dispersion. Integration over the polarization and absorption noise fields, modelled by a set of harmonic oscillators, allowed us to eliminate the matter degrees of freedom giving effective action describing dynamics of the electromagnetic field in a dispersive medium. The dielectric function, which can be determined from the effective action of the classical electromagnetic field, depends both on frequency and wave vector, and the constituent equations are therefore non-local in time and space variables.

Full quantization of the model by path integration gives expressions for the quantum propagators of the effective fields, i.e. the electromagnetic and polarization field. Poles of the propagators determine the dispersion relation of the type $\omega = \omega(k)$, which for an absorptive medium cannot be fulfilled for both frequency and wave vector real. Therefore, contrary to vacuum QED, frequency and wave vector have to be treated as independent variables, with independent integrations over both of them in plane-wave expansions of field operators. The field operators can be retrieved from the propagators assuming existence of creation and annihilation operators of effective quanta of the electromagnetic field in the medium—the polaritons. The procedure of construction of field operators from the propagators is consistent in that the effective field operators, which are 'secondary' objects in this quantization procedure, fulfil canonical commutation rules, which serve as a basic assumption in the canonical quantization approach.

Appendix A

The solution of equation (5.4) will be found by calculating the imaginary parts of both sides. Using

$$\frac{1}{x \pm \mathrm{i}\epsilon} = P\frac{1}{x} \mp \mathrm{i}\pi\delta(\mathrm{x}) \tag{A.1}$$

one obtains from (5.4), for positive frequencies ω,

$$-\frac{1}{2\hbar}|\phi(\omega,\mathbf{k})|^2 = \frac{1}{\varepsilon_0}\mathrm{Im}\left[\frac{1}{\omega^2\varepsilon(\omega,\mathrm{k}) - c^2\mathrm{k}^2}\right], \tag{A.2}$$

where $\widetilde{\varepsilon}$ was replaced by the dielectric function ε since $\widetilde{\varepsilon} = \varepsilon$ for positive frequencies. Equation (A.2) allows us to determine ϕ up to a phase factor. Using

$$\mathrm{Im}\left[\frac{1}{\omega^2\varepsilon(\omega,\mathrm{k}) - c^2\mathrm{k}^2}\right] = -\frac{\omega^2\,\mathrm{Im}[\varepsilon(\omega,\mathbf{k})]}{|\omega^2\varepsilon(\omega,\mathrm{k}) - c^2\mathbf{k}^2|^2}, \tag{A.3}$$

and choosing the phase factor as equal to unity gives

$$\phi(\omega,\mathrm{k}) = \left(\frac{2\hbar}{\varepsilon_0}\right)^{1/2}\frac{\omega\{\mathrm{Im}[\varepsilon(\omega,\mathrm{k})]\}^{1/2}}{\omega^2\varepsilon(\omega,\mathrm{k}) - c^2\mathbf{k}^2}. \tag{A.4}$$

Although expression for the vector potential operator involves the function ϕ only for positive frequencies, it is nevertheless important to know its explicit form over the entire frequency axis. This can be achieved by assuming that $|\phi|^2$ considered as a function of the frequency over the entire real axis is odd [5, 36]. For negative frequencies one gets from (5.4)

$$-\frac{1}{2\hbar}|\phi(-\omega,\mathbf{k})|^2=\frac{1}{\varepsilon_0}\mathrm{Im}\left[\frac{1}{\omega^2\widetilde{\varepsilon}(\omega,\mathbf{k})-c^2\mathrm{k}^2}\right]. \tag{A.5}$$

Since for negative frequencies $\widetilde{\varepsilon}=\varepsilon^*$, equation (A.5) leads to (A.4) also for negative semi-axis. The odd character of $|\phi|^2$ is consistent with the property $\mathrm{Im}[\varepsilon(\omega,\mathbf{k}]=-\mathrm{Im}[\varepsilon(-\omega,\mathbf{k}]$.

Equations (5.7*a*) and (5.7*b*) for the functions p and q, and also equation for the function f determining scalar potential, can be solved in a similar way with the only difference that the phase factors cannot be put equal to unity, but have to be determined from Maxwell equations. What really counts is the relative phase between ϕ and p, and q and f. Also for these functions the assumption that extension to negative frequencies gives odd functions is made. For positive frequencies equation (5.7*a*) gives

$$\frac{1}{2\hbar}|p(\omega,\mathbf{k})|^2=\mathrm{Im}\left[\widetilde{\Gamma}(\omega,\mathbf{k})\frac{\omega^2-c^2\mathbf{k}^2}{\omega^2\varepsilon(\omega,\mathbf{k})-c^2\mathbf{k}^2}\right]. \tag{A.6}$$

It follows from (3.23) that $\widetilde{\Gamma}(\omega,\mathbf{k})=\varepsilon_0[\varepsilon(\omega,\mathbf{k})-1]$ for positive frequencies, which upon substitution into (A.6) gives after a simple calculation

$$|p(\omega,\mathbf{k})|^2=2\hbar\varepsilon_0(\omega^2-c^2\mathbf{k}^2)^2\frac{\mathrm{Im}[\varepsilon(\omega,\mathbf{k})]}{|\omega^2\varepsilon(\omega,\mathbf{k})-c^2\mathbf{k}^2|^2}, \tag{A.7}$$

so that

$$p(\omega,\mathbf{k})=(2\hbar\varepsilon_0)^{1/2}(\omega^2-c^2\mathbf{k}^2)\frac{\{\mathrm{Im}[\varepsilon(\omega,\mathbf{k})]\}^{1/2}}{\omega^2\varepsilon(\omega,\mathbf{k})-c^2\mathbf{k}^2}\,\mathrm{e}^{\mathrm{i}\alpha}=\varepsilon_0\frac{\omega^2-c^2\mathbf{k}^2}{\omega}\,\mathrm{e}^{\mathrm{i}\alpha}\phi(\omega,\mathbf{k}). \tag{A.8}$$

Similar calculation gives for q

$$q(\omega,\mathbf{k})=(2\hbar\varepsilon_0)^{1/2}\frac{\{\mathrm{Im}[\varepsilon(\omega,\mathbf{k})]\}^{1/2}}{\varepsilon(\omega,\mathbf{k})}\,\mathrm{e}^{\mathrm{i}\beta}, \tag{A.9}$$

and

$$f(\omega,\mathbf{k})=\frac{1}{\varepsilon_0|\mathbf{k}|}q(\omega,\mathbf{k})\,\mathrm{e}^{\mathrm{i}\gamma}. \tag{A.10}$$

The phases will be determined from the equation

$$\frac{1}{\mu_0}\nabla\times\mathbf{B}=\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}+\frac{\partial\mathbf{P}}{\partial t}, \tag{A.11}$$

which in terms of the potentials reads

$$-\nabla^2\mathbf{A}=-\frac{1}{c^2}(\ddot{\mathbf{A}}+\nabla\dot{\phi})+\mu_0\dot{\mathbf{P}}. \tag{A.12}$$

After substituting Fourier transforms of positive frequency parts of field operators and separating transverse and longitudinal parts on both sides one gets

$$\left(\mathbf{k}^2-\frac{\omega^2}{c^2}\right)\phi=-\mathrm{i}\omega\mu_0 p,\qquad f+\frac{\mathrm{i}}{\varepsilon_0|\mathbf{k}|}q=0. \tag{A.13}$$

Use of (A.8) and (A.10) leads to the conditions

$$\mathrm{i}\,\mathrm{e}^{\mathrm{i}\alpha}=1,\qquad \mathrm{e}^{\mathrm{i}\gamma}=-\mathrm{i}. \tag{A.14}$$

so that $\alpha = \gamma = -\pi/2$. The phase β of the q-function will be chosen in the same way as that for p. Finally,

$$p(\omega, \mathbf{k}) = -\mathrm{i}\varepsilon_0 \frac{\omega^2 - c^2\mathbf{k}^2}{\omega}\phi(\omega, \mathbf{k}), \tag{A.15}$$

and

$$q(\omega, \mathbf{k}) = -\mathrm{i}\varepsilon_0 \left(\frac{2\hbar}{\varepsilon_0}\right)^{1/2} \frac{\{\mathrm{Im}[\varepsilon(\omega, \mathbf{k})]\}^{1/2}}{\varepsilon(\omega, \mathbf{k})}, \qquad f(\omega, \mathbf{k}) = -\frac{\mathrm{i}}{\varepsilon_0|\mathbf{k}|}q(\omega, \mathbf{k}). \tag{A.16}$$

These relations are valid over the entire frequency axis.

Appendix B

In this appendix the commutation relation (4.1) will be verified directly with the use of field operators. Using expressions for the positive frequency parts (5.10*a*) and (5.12), calculating the operators in configuration space with the use of $A_i^{(-)} = A_i^{(+)\dagger}$ (also for D_j) and using (5.3), one arrives at the following expression for the commutator

$$[A_i(t, \mathbf{r}), D_j(t, \mathbf{r}')] = -2\mathrm{i}\varepsilon_0 \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{|\phi(\omega, \mathbf{k})|^2}{\omega} c^2\mathbf{k}^2 \left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right). \tag{B.1}$$

Putting $\omega = 0$ in (5.4) and changing the integration variable from ω' to ω gives the following sum rule for the ϕ function

$$\int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{|\phi(\omega, \mathbf{k})|^2}{\omega} = \frac{\hbar}{2\varepsilon_0 c^2\mathbf{k}^2}, \tag{B.2}$$

which after substituting into (B.1) leads to the commutation relation (4.1).

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